ISRAEL JOURNAL OF MATHEMATICS 97 (1997), 221–238

PROJECTING THE ONE-DIMENSIONAL SIERPINSKI GASKET

BY

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ABSTRACT

Let $S \subset \mathbb{R}^2$ be the Cantor set consisting of points (x, y) which have an expansion in negative powers of 3 using digits $\{(0,0),(1,0),(0,1)\}$. We show that the projection of S in any irrational direction has Lebesgue measure 0. The projection in a rational direction p/q has Hausdorff dimension less than 1 unless $p+q \equiv 0 \bmod 3$, in which case the projection has nonempty interior and measure $1/q$. We compute bounds on the dimension of the projection for certain sequences of rational directions, and exhibit a residual set of directions for which the projection has dimension 1.

1. Introduction

Let S be the set of points in $\mathbb{R}^2$ with an expansion in base 3 using negative powers of the base and digits $\{(0,0),(1,0),(0,1)\}$, that is

$$S = \Big\{ \sum_{i=1}^{\infty} \alpha_i 3^{-i} \;\Big|\; \alpha_i \in \{(0,0),(1,0),(0,1)\} \Big\}.$$

* This work was partially completed while the author was at the Institut Fourier, Grenoble, France.

Received January 25, 1995

The set S is also the attractor in $\mathbb{R}^2$ for the three contracting linear maps

$$\begin{aligned} f_1\colon (x,y) &\mapsto \left(\frac{x}{3},\frac{y}{3}\right), \\ f_2\colon (x,y) &\mapsto \left(\frac{x+1}{3},\frac{y}{3}\right), \\ f_3\colon (x,y) &\mapsto \left(\frac{x}{3},\frac{y+1}{3}\right), \end{aligned}$$

by which we mean S is the (unique) minimal compact set for which $f_i(S) \subset S$ for $i = 1,2,3$.

A third description of S is the subset

$$S = \{(x,y) \in C \times C \mid x + y \in C\},$$

where C is the usual "middle third" Cantor set constructed on the interval $[0,1/2]$. See Figure 1.

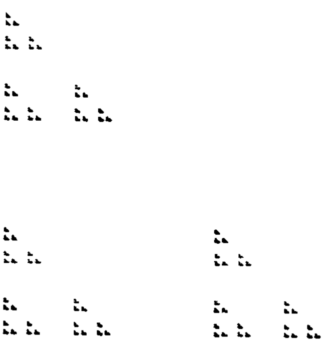

Figure 1. The set S.

We leave the reader to show that these three descriptions are all of the same set S. Since the set S is self-similar and satisfies the open set condition [3], the Hausdorff dimension of S is the number δ such that $3 \cdot (1/3)^\delta = 1$, that is, $\delta = 1$. Since S resembles the Sierpinski gasket, (defined by replacing all occurrences of 3 with 2 in the first two definitions above) we have called S the **one-dimensional Sierpinski gasket**.

We shall be concerned not so much with S as its various linear projections: define S_u to be the linear projection of S onto the x-axis, $S_u = \pi_u(S)$, where π_u sends $(0,1)$ to the point $(u,0)$, that is,

$$\pi_u = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix}.$$

For example, note that $S_0 = C$, the middle third Cantor set on the interval $[0, 1/2]$, and $S_{1/2}$ is the entire interval $[0, 1/2]$. The question is, what happens for other values of u?

LEMMA 1: *The set $S_u \subset \mathbb{R}$ is the attractor for the three linear maps*

$$\begin{aligned} x &\mapsto \frac{x}{3}, \\ x &\mapsto \frac{x+1}{3}, \\ x &\mapsto \frac{x+u}{3}. \end{aligned}$$

Alternatively it is the set of real numbers which have an expansion in base 3 *using negative powers of* 3 *and digits* $\{0, 1, u\}$.

Proof: Use the second description of S and the fact that the homothety $x \mapsto x/3$ commutes with the linear projection π_u. ∎

We shall be interested in the linear measure of the various projections S_u. A theorem of Besicovich [1] states that the projection in almost every direction of an irregular set of Hausdorff dimension 1 in $\mathbb{R}^2$ has Lebesgue 1-dimensional measure 0. We compute (Theorem 2) the measure of S_u for every u. This answers a question of Odlyzko [9]. This is also the first nontrivial example of a dynamically defined set all of whose projected measures can be computed explicitly.

Secondly, we shall be concerned with the dimension of S_u. From a theorem of Marstrand [8], almost every linear projection of a set of dimension 1 in $\mathbb{R}^2$ has dimension 1. Let $\varphi(u)$ denote the dimension of S_u. Furstenberg has conjectured that $\varphi(u) = 1$ for every irrational u. A definitive answer regarding the dimension of S_u for every u is still waiting; we give here (Theorem 13) a lower bound on $\varphi(u)$ in terms of the approximations of u by rationals. This allows us to exhibit a residual set on which $\varphi(u) = 1$.

We give an algorithm for calculating $\varphi(u)$ for any rational value of u. The algorithm allows us to give upper bounds for some sequences of rational values of u, most notably an upper bound for $\varphi(3^k p/q)$ in terms of p and q.

For the format of the paper, Section 2 deals with the measure of the set S_u. Section 3 gives an algorithm for computing $\varphi(u)$ for u rational, in Section 4 we construct a residual set on which $\varphi(u) = 1$, and in Section 5 we give a lower bound on $\varphi(u)$ when u is close to a rational. The results in these last three sections arose from attempts to prove or disprove Furstenberg's conjecture that $\varphi(u) = 1$ for all irrational u.

Notationally, throughout the paper we denote the cardinality of a set A by $|A|$. If $A \subset \mathbb{R}$ and $x \in \mathbb{R}$, by $x + A$ we mean $\{x + a \mid a \in A\}$, and by xA we mean $\{xa \mid a \in A\}$. Let μ denote 1-dimensional Lebesgue measure.

Let $\varphi(u) = \dim(S_u)$ and let $d = d_u$ denote the diameter of S_u ($d = u/2$ if $u > 1$ or $u < 0$, and $d = 1/2$ if $0 < u < 1$).

Remark: Since this paper was written, J. Lagarias and Y. Wang have found a proof of (a more general result than) Lemmas 4 and 5 below [7], using the Fourier transform of band-limited functions.

2. The measure of S_u

THEOREM 2: *$\mu(S_u) > 0$ iff u is a rational of the form p/q in lowest terms with $p + q \equiv 0 \bmod 3$. Also $\varphi(u) < 1$ if u is rational of the form p/q with $p + q \not\equiv 0 \bmod 3$.*

The proof of this theorem follows from 4 lemmas:

LEMMA 3: *If $\mu(S_u) > 0$ then S_u contains intervals.*

LEMMA 4: *If S_u contains an interval then u is rational.*

LEMMA 5: *If $u = p/q$ in lowest terms with $p + q \equiv 0 \bmod 3$ then $\mu(S_u) = 1/q$.*

LEMMA 6: *If $u = p/q$ in lowest terms with $p + q \not\equiv 0 \bmod 3$ then $\varphi(u) < 1$.*

Proof of Lemma 3: Suppose $\mu(S_u) > 0$. From the definition of S_u we have $S_u \cup (S_u+1) \cup (S_u+u) = 3S_u$. Thus $3\mu(S_u) \geq \mu(S_u \cup S_u+1 \cup S_u+u) = \mu(3S_u) = 3\mu(S_u)$, the last equality from the scaling property of Lebesgue measure. Thus the three translated copies of S_u which cover $3S_u$ must be disjoint in measure. Similarly for any $n \geq 0$, $3^n S_u$ is covered by 3^n translates of S_u pairwise disjoint in measure.

Let x be a Lebesgue point of S_u (a point of density for Lebesgue measure).

Among the 3^n translates of S_u which cover $3^n S_u$, let $S_u+v_1, S_u+v_2, \ldots, S_u+v_k$ be those translates that intersect the interval $[3^n x - 1, 3^n x + 1]$.

Let $V_n = \bigcup_{i=1}^k S_u + v_i$ be the union of these translates, and $U_n = \{v_1 - 3^n x, \ldots, v_k - 3^n x\}$ the translations relative to $3^n x$.

Since the translates of S_u are disjoint in measure and have positive measure, there is a bound independent of n on $k = |U_n|$. Furthermore, the size of an element of U_n is at most $\operatorname{diam}(S_u) + 1$.

Since U_n is a set of bounded real numbers and has bounded cardinality independent of n, there is a sequence of integers $\{n_i\}$ such that the U_{n_i} converge, that is the cardinality is eventually constant and each element of U_{n_i} converges.

Let U be the limiting set of translations, $U = \lim_{n\to\infty} U_{n_i}$. There is a corresponding arrangement $V = \bigcup_{v\in U} S_u + v$.

Since x was a Lebesgue point, the measure of $3^n S_u \cap [3^n x - 1, 3^n x + 1] = V_n \cap [3^n x - 1, 3^n x + 1]$ converges to 2. Since the S_u are closed, V contains an entire interval of length 2.

By the Baire category theorem, if a countable (in this case finite) number of closed sets covers an open set, then at least one of the closed sets contains an open set. So one of the S_u contains an interval. ∎

The proof of Lemma 4 relies on the following.

LEMMA 7: *If S_u has interior, then it is the closure of its interior. For all u, the boundary of S_u has measure 0.*

Proof: Suppose S_u has interior. Let $y \in S_u$, with $y = \sum_{i=1}^{\infty} \alpha_i 3^{-i}$. Then for each $k \geq 0$, y is in the set $(\sum_{i=1}^{k} \alpha_i 3^{-i}) + 3^{-k} S_u$, whose diameter is $3^{-k}\operatorname{diam}(S_u)$, and has interior. Thus y is a limit of points in the interior of S_u.

If $\mu(\partial S_u) > 0$, then $\mu(S_u) > 0$ so that S_u contains an interval. Then for n large $3^n S_u$ contains a large interval I which is covered by translates of S_u pairwise disjoint in measure. But the boundary of one of the translates strictly contained in I is contained in the union of the boundaries of the other translates of S_u. Thus the translates overlap in measure, a contradiction. ∎

Proof of Lemma 4: Suppose S_u contains an interval $[a, b]$, with $a < b$.

As before, $3^n S_u$ is covered by 3^n translated copies of S_u which are disjoint in measure, hence have nonoverlapping interiors. We shall refer to these translated copies of S_u as **tiles**. Since $0 \in S_u$, $v \in S_u + v$ and we shall refer to v as the **center** of the tile $S_u + v$.

Let n be large enough so that $3^n|b - a| > 3\operatorname{diam}(S_u)$. The interval $3^n[a, b] \subset 3^n S_u$ is tiled by translates of S_u. Let S, S' be two of these tiles which are contained entirely in $3^n[a, b]$. We suppose that the center of S' is right of the center of S.

Now the set $3^n S$ contains an interval $[a_1, b_1]$ which is a translate of the interval $3^n[a,b] \subset 3^n S_u$. Similarly $3^n S'$ contains such an interval $[a_2, b_2]$. In particular each of these three intervals is tiled in the same way, that is, has the same relative translations of tiles with respect to the center of the interval.

There is a unique way to extend the tiling of $[a_1, b_1]$ to a tiling of the half-line $[a_1, \infty)$ using translates of S_u as tiles: let V be the union of tiles of $3^{2n} S_u \supset 3^n S$ intersecting $[a_1, b_1]$; let I be the maximal interval in V containing $[a_1, b_1]$. Let y be the right endpoint of I; in any tiling of $[a_1, \infty)$ extending the given one there must be a tile whose leftmost point is the point y (recall from Lemma 7 that the tiles S_u are closures of their interior). Adding this tile to V increases the length of the maximal interval I, so the next tile is determined, and so on. Each tile added adds length at least $b - a$ to I, and so a tiling of $[a_1, \infty)$ is determined.

Since $[a_1, b_1]$ and $[a_2, b_2]$ were both already contained in the tiling of $3^{2n}[a,b]$ determined by $3^{2n} S_u$, the unique tiling of $[a_1, \infty)$ extending that in $[a_1, b_1]$ contains the tiling already present in $[a_2, b_2]$. But since the arrangement in $[a_2, b_2]$ is the same as that in $[a_1, b_1]$, the tiling of the ray $[a_1, \infty)$ repeats itself, that is, it is periodic with period $a_2 - a_1$.

The same argument works in the other direction starting from $[a_2, b_2]$ showing that the unique tiling of the whole line determined by $[a_1, b_1]$ is periodic. Let T be this tiling.

If we take this periodic tiling T, multiply each tile $S + x$ by 3 and subdivide it into its three parts $S + 3x, S + 3x + 1, S + 3x + u$, we have a new tiling T' which contains the same subtiling in $[a_1, b_1]$. This implies T' is the *same* tiling as T. Thus for any $n \geq 0$ each tile in $3^n S_u$ is a tile in T.

Since T is periodic, the centers of tiles are all contained in a set of the form $A + \alpha\mathbb{Z}$, where $\alpha \in \mathbb{R}$, $\alpha > 0$ and A is a finite subset of $[0, \alpha)$.

But there exist tiles in $3^n S_u$ centered at all the points $1, 3, 3^2, \ldots, 3^{n-1}$ and also at the points $u, 3u, 3^2 u, \ldots, 3^{n-1} u$. Since A is finite we must have for some m, m' with $m \neq m'$ that $3^m - 3^{m'} \in \alpha\mathbb{Z}$, so that α is rational. But also there are r, r' with $r \neq r'$ such that $3^r u - 3^{r'} u \in \alpha\mathbb{Z}$ which (since α is rational) implies that u is rational. ∎

Since we will use this last result in the proof of Lemma 5 we set it aside as a corollary:

COROLLARY 8: *If $u = p/q$ with $p + q \equiv 0 \bmod 3$ then there is a tiling of $\mathbb{R}$ by* *translates of qS_u, with integer period, such that for each n the tiling of $3^n qS_u$ by translates of qS_u is a subtiling.*

Proof of Lemma 5: Let $u = p/q$ in lowest terms, with $p, q \in \mathbb{Z}$ and $p + q \equiv 0 \bmod 3$. We also assume without loss of generality that $0 < p < q$. Let $S = S_{p,q} \stackrel{\text{def}}{=} \{\sum_{i=1}^{\infty} \alpha_i 3^{-i} \mid \alpha_i \in \{0, p, q\}\}$, so that $S = qS_u$. We will show that S has Lebesgue measure 1, so that $S_{p/q}$ has measure $1/q$.

Define

$$S^n = \left\{ \sum_{k=1}^{n} \alpha_k 3^{-k} \mid \alpha_k \in \{0, p, q\} \right\}.$$

Then S^n consists of 3^n triadic rationals, which are distinct by the condition on p and q: if

$$\sum_{k=1}^{n} \alpha_k 3^{-k} = \sum_{k=1}^{n} \beta_k 3^{-k}$$

then

$$\sum_{k=1}^{n} \alpha_k 3^{n-k} = \sum_{k=1}^{n} \beta_k 3^{n-k},$$

and this is an integer; taking this equation modulo 3 gives $\alpha_n \equiv \beta_n \bmod 3$ which implies $\alpha_n = \beta_n$. Taking the equation modulo 9 then yields $\alpha_{n-1} = \beta_{n-1}$, and so on.

Define probability measures $\mu_n = 3^{-n} \sum_{x \in S^n} \delta(x)$ where $\delta(x)$ is the point mass of mass 1 at x.

The measure μ_n is a sum of point masses of mass 3^{-n} at distinct points of the lattice $3^{-n}\mathbb{Z}$. Thus for each triadic interval

$$I = \left[\frac{p}{3^k}, \frac{p+1}{3^k} \right) \quad \text{with } k \leq n$$

we have $\mu_n(I) \leq \mu(I)$ (μ is Lebesgue measure). Hence any weak limit μ_∞ of the μ_n satisfies $\mu_\infty(I) \leq \mu(I)$ for any interval I. Since S is closed, $\mu(S)$ is approximated by finite collections of intervals, so we have $1 = \mu_\infty(S) \leq \mu(S)$.

To show that $\mu(S) = 1$, we will show that the periodic tiling given by Corollary 8 is in fact periodic with period 1.

Let $R \in \mathbb{Z}_+$ be the period of the tiling. Then there is a set $W \subset [0, R) \cap \mathbb{Z}$ such that each tile is of the form $x + S$, where $x \in W + R\mathbb{Z}$, and there are tiles at each point of $W + R\mathbb{Z}$. The invariance of the tiling under expansion and subdivision of the tiles implies that the set W taken modulo R is invariant under the three maps $x \mapsto 3x, x \mapsto 3x+p, x \mapsto 3x+q$. We will show that this implies $W = [0,R) \cap \mathbb{Z}$.

Let G be the directed graph with vertices $V = [0,R) \cap \mathbb{Z}$ and edges from x to $(3x+d) \bmod R$ for each $d \in \{0,p,q\}$. We show that G is strongly connected. Let $f\colon V \to \mathbb{R}$ be an eigenvector for the adjacency matrix T with eigenvalue 3: $Tf(x) = f(3x) + f(3x+p) + f(3x+q) = 3f(x)$. It suffices to show that f is constant on V.

Identify V with $\mathbb{Z}/R\mathbb{Z}$. We expand f using the characters $\{\chi_n\}$ of $\mathbb{Z}/R\mathbb{Z}$: the nth Fourier coefficient of f is

$$\chi_n(f) = \sum_{k=0}^{R-1} f(k) e^{2\pi i n k/R}.$$

Assume first that 3 does not divide R. Then

$$\begin{aligned}\chi_n(3f)) = \chi_n(Tf) &= \sum_{k=0}^{R-1} Tf(k) e^{2\pi i n k/R} \\ &= \sum_{k=0}^{R-1} (f(3k) + f(3k+p) + f(3k+q)) e^{2\pi i n k/R} \\ &= \sum f(3k) e^{2\pi i n k/R} + f(3k) e^{2\pi i n (k-p)/R} + f(3k) e^{2\pi i n (k-q)/R} \\ &= \chi_n(f(3x)) \cdot (1 + e^{-2\pi i n p/R} + e^{-2\pi i n q/R}).\end{aligned}$$

Now since $3^{\phi(R)} \equiv 1 \bmod R$, we have

$$3^{\phi(R)} \chi_n(f) = \chi_n(T^{\phi(R)} f) = \chi_n(f) \cdot \prod_{k=0}^{\phi(R)-1} (1 + e^{-2\pi i 3^k n p/R} + e^{-2\pi i 3^k n q/R}).$$

If $\chi_n(f)$ is nonzero, each term in the product must have modulus 3. This implies $e^{2\pi i n p/R} = 1$ and $e^{2\pi i n q/R} = 1$, so that $np/R \in \mathbb{Z}$ and $nq/R \in \mathbb{Z}$. Since p and q are relatively prime, there exists $a, b \in \mathbb{Z}$ such that $ap + bq = 1$ and so $anp/R + bnq/R = n/R \in \mathbb{Z}$. Thus $\chi_n(f)$ is nonzero only if $n = 0$. This implies that f is constant, and so W, the support of f, is $[0,R) \cap \mathbb{Z}$.

When 3 divides R, say $R = 3^k \cdot a$ with $3 \nmid a$, then, using $3^{\phi(a)} 3^k x \equiv 3^k x \bmod R$, we have

$$\begin{aligned}3^{\phi(a)} \chi_n(f(3^k x)) &= \chi_n(T^{\phi(a)} f(3^k x)) \\ &= \chi_n(f(3^k x)) \prod_{j=0}^{\phi(a)} (1 + e^{-2\pi i 3^j n p/R} + e^{-2\pi i 3^j n p/R})\end{aligned}$$

which implies as before that $\chi_n(f(3^k x)) = 0$ unless $n = 0$.

But now

$$3^k \chi_n(f(x)) = \chi_n(T^k f(x)) = \chi_n(f(3^k x)) \prod_{j=0}^{k} (1 + e^{-2\pi i 3^j np/R} + e^{-2\pi i 3^j np/R})$$

and so $\chi_n(f) = 0$ for all $n \neq 0$.

Hence in each case $W = [0, R) \cap \mathbb{Z}$ and so the tiling is periodic with period 1. Thus the measure of $qS(u)$ is exactly 1. ∎

Proof of Lemma 6: Define S^n as in the previous proof, using digits $\{0, p, q\}$.

If for some n, S^n has fewer than 3^n elements, say $|S^n| = k < 3^n$, then $3^n S$ can be covered by k translates of S, and therefore the dimension of S is at most $\frac{1}{n}\log_3(k) < 1$.

Otherwise, since again $S^n \subset 3^{-n}\mathbb{Z}$, the measure μ_∞ constructed as in the previous proof is absolutely continuous with respect to Lebesgue measure, and the projection μ'_∞ of μ_∞ to $\mathbb{R}/\mathbb{Z}$ is an invariant measure for the circle endomorphism $T\colon z \mapsto 3z$ (note that for each n, $T_*\mu'_n = \mu'_{n-1}$ so that $T_*\mu'_\infty = \mu'_\infty$).

Since the map $z \mapsto 3z$ is ergodic the only invariant measure absolutely continuous with respect to Lebesgue measure is Lebesgue measure itself. So μ'_∞ is a multiple of Lebesgue measure.

The Fourier coefficients $\hat{\mu}'_\infty(k)$ are thus all zero except for $\hat{\mu}'_\infty(0)$. But

$$\hat{\mu}'_\infty(k) = \prod_{j=1}^{\infty} \frac{1}{3}(1 + e^{2\pi i pk3^{-j}} + e^{2\pi i qk3^{-j}}),$$

and if this is zero then some term $(1 + e^{2\pi i kp3^{-j}} + e^{2\pi i qk3^{-j}})$ in the product must be zero (the terms converge exponentially fast towards 1). A set of three complex numbers of modulus 1 whose sum is zero must be a rotation of the set $\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. This implies that

$$kp3^{-j} \equiv \frac{1}{3} \bmod \mathbb{Z},$$
$$kq3^{-j} \equiv \frac{2}{3} \bmod \mathbb{Z},$$

or vice versa (exchanging p and q). Multiplying by 3 gives

$$kp3^{-j+1} \equiv 1 \bmod 3\mathbb{Z},$$
$$kq3^{-j+1} \equiv 2 \bmod 3\mathbb{Z}.$$

Find integers a, b such that $ap + bq = 1$; then multiplying the first equation by a and the second by b and adding we find $k3^{-j+1} \in \mathbb{Z}$. Furthermore, either equation gives $k3^{-j+1} \not\equiv 0 \bmod 3$. Now simply summing the two equations gives $p + q \equiv 0 \bmod 3$, a contradiction. ∎

3. The dimension of S_u in rational directions

Define $S_u^k = \{\sum_{i=1}^{k} \alpha_i 3^{-i} \mid \alpha_i \in \{0, 1, u\}\}$.

PROPOSITION 9: *Let $u = p/q$. Then $\lim_{k\to\infty} |S_u^k|^{1/k}$ is the Perron eigenvalue of a nonnegative integer matrix.*

Proof: Assume that $0 < p < q$. We compute the cardinality of $3^n S^n \subset \mathbb{Z}$. For a sequence $\{x_1, x_2 \ldots, x_n\}$ of digits $\{0, p, q\}$ define

$$R(\{x_1, x_2, \ldots, x_n\}) = \sum_{i=1}^{n} x_i 3^i \in \mathbb{Z}$$

to be the corresponding integer in $3^n S^n$.

For each sequence $\{x_1, \ldots, x_n\}$ there is a *first* sequence $\{y_1, \ldots, y_n\}$ (of the same length) in lexicographic order which yields the same point in $3^n S^n$, that is $R(\{y_1, \ldots, y_n\}) = R(\{x_1, \ldots, x_n\})$. (Lexicographic order means $\{y_1, \ldots, y_n\} < \{x_1, \ldots, x_n\}$ iff on the first index in which $y_i \neq x_i$ we have $y_i < x_i$.)

Let T_n be the set of all such first sequences of length n, so that elements of $R(T_n)$ are distinct integers and $R(T_n) = 3^n S^n$.

Let $m = \lfloor \frac{q-1}{2} \rfloor$. Let G be a graph with $2m + 1$ vertices labelled with the integers from $-m$ to m inclusive, and edges as follows. Each edge of G is labelled with an ordered pair of digits $\{0, p, q\}$. The vertex 0 has an outgoing edge labelled (d_1, d_2) for every pair for which $d_1 > d_2$ and $(d_1 - d_2)/3$ is an integer. This edge terminates at the vertex labelled $(d_1 - d_2)/3$. For every vertex $x \neq 0$ and pair of digits (d_1, d_2), if $(x + d_1 - d_2)/3$ is an integer put an edge with label (d_1, d_2) from the vertex x to the vertex $(x + d_1 - d_2)/3$. (See illustration after the proof.)

Now we claim that T_n consists of all words of length n such that no substring is the set of first coordinates of edge labels on a path in G from vertex 0 to vertex 0. That is, if $(d_{i_1}, d_{j_1}), \ldots, (d_{i_n}, d_{j_n})$ is the sequence of edge labels on a path from 0 to 0 in G then no element of T contains the substring $d_{i_1}, \ldots, d_{i_n}$, and conversely. We abbreviate this concept by saying no substring **labels a path** in G.

To see this, let γ be a path in the graph G from vertex 0 to vertex 0, with edge labels $\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$. Then from the definition of the edge labels

$$\frac{\dfrac{\dfrac{\frac{x_1-y_1}{3}+x_2-y_2}{3}+x_3-y_3}{\ddots} + \ldots + (x_k - y_k)}{3} = 0$$

so that $R(\{x_1, \ldots, x_n\}) = R(\{y_1, \ldots, y_n\})$. In addition we must have that $y_1 < x_1$ since γ starts from 0, so that the sequence $\{y_i\}$ is lower in lexicographic order than $\{x_i\}$, so $\{x_1, \ldots, x_k\}$ is not a substring of any sequence in T_n.

Conversely if a sequence $\{x_1, \ldots, x_n\}$ has a lower sequence $\{y_1, \ldots, y_n\}$ with $R(\{x_i\}) = R(\{y_i\})$ and $x_1 \neq y_1$ then the sequence $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ labels a path from 0 to 0 in G.

This proves the claim. Now the problem is reduced to counting the sequences in $\{0, p, q\}^*$ such that no substring labels a path from 0 to 0 in the graph G. This problem is well-known and not difficult [2]; the set of such sequences is a regular language. ∎

COROLLARY 10: *For u rational, $\varphi(u) = \lim_{k\to\infty} \frac{1}{k} \log_3 |S_u^k|$.*

Proof: $|S_u^k|$ is easily seen to be the number of triadic intervals of size 3^{-k} needed to cover S. From the proposition, $\lim \frac{1}{k} \log_3 |S_u^k|$ exists, and hence is the Minkowski dimension (i.e. box dimension) of S. But Falconer [4] has shown that for self-similar sets (attractors of similarities) the Minkowski and Hausdorff dimensions agree. ∎

Keane and Smorodinsky [6] showed that $\varphi(3) = \log_3(\frac{1+\sqrt{5}}{2})$. Yuval Peres generalized their technique to yield $\varphi(3^n) = \log_3(\frac{1+\sqrt{5}}{2})$ for all $n \geq 1$.

Example: Let $u = 1/6$, so that the digits are $\{0, 1, 6\}$. The graph G constructed in the above proof is shown in Figure 2.

Now let $u = 1/(2 \cdot 3^k)$. Then the graph $G = G_u$ contains the subgraph shown in Figure 3, so that the set of disallowed substrings includes all strings of the form

$$u, \underbrace{*, *, \ldots, *}_{k-1}, 1, 0,$$

where the $*$'s can each be any digit. There are 3^{k-1} such strings of length $k + 2$.

To obtain an upper bound on the number of strings of length $n > k+2$ which don't contain a substring of the form $u, \underbrace{*, *, \ldots, *}_{k-1}, 1, 0$, divide a string of length n into $\lfloor \frac{n}{2k} \rfloor$ substrings, each of length $2k$, plus a leftover string of length $n - 2k\lfloor \frac{n}{2k} \rfloor$. There are 3^{2k} possible strings of length $2k$; but 1/27th of these have a disallowed word occurring at the first position. Of the remaining $3^{2k} \cdot 26/27$ words, 1/27th have a disallowed word occurring at position 3, and so on. The occurrences of disallowed subwords at positions $1, 3, 5, \ldots, k-1$ (if k is even) are independent. Thus there are at most $3^{2k}(26/27)^{k/2}$ allowed words of length $2k$. Thus the number of allowed strings of length n is at most

$$\left(3^{2k}\left(\frac{26}{27}\right)^{k/2}\right)^{\lfloor n/2k \rfloor} \cdot 3^{n-2k\lfloor \frac{n}{2k} \rfloor} \approx 3^n \left(\frac{26}{27}\right)^{n/4}$$

if n is large. Thus the dimension of S_u is at most $1 - \frac{1}{4}\log_3(27/26)$.

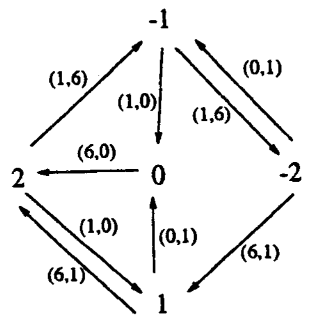

Figure 2.

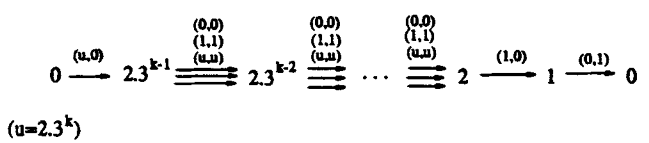

Figure 3. A subgraph of $G(2 \cdot 3^k)$.

A similar argument for $u = p/q$ yields:

THEOREM 11: *If $u = p/q$ in lowest terms, $0 < p < q$, $3 \nmid q$ and $k > 0$, then there is an upper bound $\varphi(3^k u) < f_{p,q} < 1$ independent of the integer k. Explicitly, if $k > 2\log_3(2q)$ then*

$$\varphi\left(\frac{3^k p}{q}\right) < 1 - \frac{1}{16pq\log(2q)}.$$

Proof: Construct the graph $G = G_u$ as in the proof of Proposition 9 using digits $\{0, q, 3^k p\}$ which have no factor in common.

Represent q in base 3 using digits $\{0, 1, -1\}$:

$$q = \sum_{i=0}^{\ell} \alpha_i 3^i, \qquad \text{where } \alpha_i \in \{0, 1, -1\}, \quad \alpha_\ell \neq 0.$$

(It is well known and easy to check that there exists a unique expansion of this form.) Note that $3^\ell/2 < q < 3^{\ell+1}/2$. Similarly, represent p in base 3 with digits $\{0, 1, -1\}$:

$$p = \sum_{i=0}^{\ell'} \beta_i 3^i, \qquad \text{where } \beta_i \in \{0, 1, -1\}, \quad \beta_{\ell'} \neq 0.$$

Then $3^{\ell'}/2 < p \leq 3^{\ell'+1}/2$, and $qp = \sum_{i=0}^{\ell'} q\beta_i 3^i$. Assume that $\alpha_1 = 1$ (for $\alpha_1 = -1$ a symmetric argument holds).

Let $k > \ell$. Then there is a path in G leading from vertex 0 to vertex $pq \cdot 3^{k-\ell}$: it is the path whose ith edge label is $(3^k p, 0)$ if $\alpha_i = 1$, $(0, 3^k p)$ if $\alpha_i = -1$, and $(0,0)$ if $\alpha_i = 0$.

There are $3^{k-\ell}$ paths in G leading from vertex $pq3^{k-\ell}$ to vertex pq (these paths go through vertices $3^{k-\ell-j}pq$, for $j \in [0, k-\ell]$) in which any of three edge labels $(0,0), (q,q), (3^k p, 3^k p)$ may be employed.

Similarly there is a path from pq to 0 whose ith edge label is $(0, q)$ if $\beta_i = 1$, $(q, 0)$ if $\beta_i = -1$, and $(0,0)$ if $\beta_i = 0$.

The paths from 0 to 0 which are concatenations of these three parts each have total length $\ell + k - \ell + \ell' = k + \ell'$. Thus there are at least $3^{k-\ell}$ disallowed strings of length $k + \ell'$ in T^n. Now by a calculation similar to the above example, the result follows: divide a string of length n into $\lfloor \frac{n}{2k} \rfloor$ strings of length $2k$. For a given string of length $2k$, accounting for the disallowed strings at positions $0, \ell, 2\ell, \ldots, \lfloor \frac{k}{\ell} \rfloor \ell$ leaves at most

$$3^{2k}(1 - 3^{-\ell-\ell'})^{\lfloor \frac{k}{\ell} \rfloor}$$

allowed strings, for a growth rate of

$$\gamma \leq 3(1 - 3^{-\ell-\ell'})^{\lfloor \frac{k}{\ell} \rfloor \frac{1}{2k}},$$

so that

$$\log_3 \gamma \leq 1 + \lfloor \frac{k}{\ell} \rfloor \frac{1}{2k} \log_3(1 - 3^{-\ell-\ell'})$$

$$< 1 - \frac{\lfloor \frac{k}{\ell} \rfloor}{2k3^{\ell+\ell'} \log 3}$$

$$< 1 - \frac{\lfloor \frac{k}{\ell} \rfloor}{8kpq \log 3}$$

$$< 1 - \frac{1}{16pq\ell \log 3} \qquad \text{if } k > 2\ell$$

$$< 1 - \frac{1}{16pq \log(2q)}.$$

Here in the second inequality we have used $\log_3(1-x) < -x/\log 3$, in the third and fifth inequality we have used $3^\ell < 2q$ and $3^{\ell'} < 2p$, and in the fourth, if $k > 2\ell$ then $\frac{1}{k}\lfloor \frac{k}{\ell} \rfloor > \frac{1}{2\ell}$. ∎

We note that the same method works for many other simple sequences of rationals: any sequence in which the graphs G have roughly 3^k cycles through 0 of length 3^k, for example $\{0, 3^n, (3^m - 1)/2\}$. It seems that all sequences with this property converge to rational values of u, though, so they cannot serve as evidence against the conjecture of Furstenberg.

4. The dimension in a residual set of directions

If two real numbers u, v are close, the corresponding sets S_u, S_v are close in the Hausdorff metric on compact sets in $\mathbb{R}$. This fact quickly leads to the following:

THEOREM 12: *Let u be a real number and p_i/q_i a sequence of rationals such that $p_i + q_i \equiv 0 \bmod 3$, $q_i \to \infty$, and such that there exists constants $C, \alpha > 0$ for which*

$$\left| u - \frac{p_i}{q_i} \right| < \frac{C}{q_i^\alpha}.$$

Then $\varphi(u) \geq 1 - 1/\alpha$.

In particular if u is super-polynomially-well approximated by rationals of the form p/q, $p + q \equiv 0 \bmod 3$, then $\varphi(u) = 1$. The set of u with this property is a residual subset of $\mathbb{R}$.

Proof: Let $N_u(\epsilon)$ be the minimum number of intervals of length ϵ needed to cover S_u. We can assume that the intervals of size ϵ are of the form $[m\epsilon, (m+1)\epsilon]$, that is, are "lattice intervals". Restricting to these intervals changes $N_u(\epsilon)$ by a constant factor and so does not affect the asymptotic growth rate of $N_u(\epsilon)$, which is the quantity we are interested in.

It suffices to show that

$$\text{if } |u - \frac{p}{q}| < \left(\frac{2}{3}\right)\frac{3^{-k}}{q}, \quad \text{then } N_u\left(\frac{3^{-k}}{q}\right) \geq \text{ const } \cdot 3^k.$$

For then, using $\epsilon = 3^{-k}/q = C/q^\alpha$, we have

$$\varphi(u) = \lim_{\epsilon\to 0}\frac{\log N_u(\epsilon)}{-\log\epsilon} = \lim_{i\to\infty}\frac{\log N_u(\frac{C}{q_i^\alpha})}{\alpha\log q_i + \text{const}} \geq \lim_{i\to\infty}\frac{\log(q_i^{\alpha-1}) + \text{const}}{\alpha\log q_i + \text{const}} = 1 - \frac{1}{\alpha}.$$

Suppose $|u - p/q| < (2/3)3^{-k}/q$. The set $S^k_{p/q} \subset S_{p/q}$ consists of 3^k distinct points on the lattice $(3^{-k}/q)\mathbb{Z}$. So it takes at least 3^k lattice intervals of length $3^{-k}q$ to cover $S_{p/q}$.

For each point $x \in S_{p/q}$, $x = \sum_{i=1}^\infty \alpha_i 3^{-i}$, $\alpha_i \in \{0, p/q, 1\}$, let x' be the point with the same sequence of digits, replacing all occurrences of the digit p/q with u. Then

$$|x - x'| < \sum_{i=1}^\infty \left|u - \frac{p}{q}\right| 3^{-i} = \frac{3}{2}\left|u - \frac{p}{q}\right| < \frac{3^{-k}}{q}.$$

So each point of $S_{p/q}$ is within $3^{-k}/q$ of a point of S_u. So it takes at least 3^{k-1} intervals of size $3^{-k}/q$ to cover S_u. Thus $N_u(3^{-k}/q) \geq 3^{k-1}$. ∎

5. Lower bounds on the dimension

The argument of the previous section can be refined to give a lower bound on $\varphi(u)$ when u is close to a single rational.

Let $u = p/q$ in lowest terms with $p + q \equiv 0 \bmod 3$ and $0 < p < q$. Let $S = qS_u$ as before.

For a sequence $\{x_1, x_2, \ldots\}$ in $\{0, p, q\}^{\mathbb{N}}$ we define

$$r(\{x_1, x_2, \ldots\}) = \sum_{i=1}^\infty x_i 3^{-i} \in S.$$

We build a graph G_∂ which describes sequences $\{x_1, x_2, \ldots\}$ for which $r(\{x_i\})$ is in the boundary ∂S of S. Let $m = \lfloor\frac{q-1}{2}\rfloor$ (we assume $q > 2$). The graph G_∂ has $2m$ vertices, labelled with the nonzero integers from $-m$ to m inclusive. From a vertex labelled x for every $d_1, d_2 \in \{0, p, q\}$ there is an edge labelled (d_1, d_2) pointing to vertex $3x + d_1 - d_2$ whenever this is a nonzero integer in the range $[-m, m]$.

Let α be an infinite path in G_∂ starting from vertex k with edge labels $(x_0,y_0),(x_1,y_1),\ldots$. Then $r(\{x_0,x_1,\ldots\})+k=r(\{y_0,y_1,\ldots\})$. Since by Lemma 5 the integer translates of S have disjoint interiors, the point $r(\{x_0,x_1,\ldots\})$ is in the boundary of S. Conversely, for a point $x\in\partial S$ there is a translate $S-k$ which also contains x since the translates of S tile $\mathbb{R}$. If $x\notin\{0,\frac{q}{2}\}$ then $|k|\le\lfloor\frac{q-1}{2}\rfloor$, so there is a $y\in S$ with $x=y-k$ and hence an infinite path in G_∂ with edge labels whose first coordinates are $\{x_i\}$ with $r(\{x_i\})=x$. (Thus G_∂ describes all boundary points of S except 0 and $q/2$.)

For each vertex v of G_∂ there is a word of length $<\log_3 q$ which does not label a path from v: if $v>0$ a word consisting entirely of qs leads successively to vertices $v_2=3v+q-d_1\ge 3v, v_3=3v_2+q-d_2\ge 3^2v,\ldots$, until $v_{\lfloor\log_3 q\rfloor}>m$. Similarly, if $v<0$ the word of length $\lfloor\log_3 q\rfloor$ consisting entirely of 0s does not label a path from v.

THEOREM 13: *Let $u\in\mathbb{R}$ and suppose there are relatively prime integers p,q, $0<p<q$, $p+q\equiv 0 \bmod 3$, and ϵ, $0<\epsilon<q^{-2q}$ such that*

$$\left|u-\frac{p}{q}\right|<\epsilon.$$

Then

$$\varphi(u)>1-\frac{1}{\frac{\log 1/\epsilon}{q\log q}-1}.$$

In particular if ϵ is small compared to q^{-2q} then $\varphi(u)$ is close to 1.

Proof of Theorem 13: We assume without loss of generality that $0<u<1$ and $0<p<q$. Let $S=qS_{p/q}$, so that $\mathrm{diam}S_u=q/2$.

For each $k>0$ let $W_k\subset\{0,p,q\}^{\mathbb{N}}$ be the set of sequences $x=\{x_1,x_2,\ldots\}$ such that for each $\ell\ge 0$, $r(\sigma^\ell x)$ is at distance at least $\frac{q}{2}3^{-k}$ from the boundary of S. (Here σ is the left shift, $\sigma^\ell(\{x_1,\ldots\})=\{x_{\ell+1},x_{\ell+2},\ldots\}$.)

An element of W_k is characterized by the property that no substring of length k labels a path in the graph G_∂ described above. For if $x_\ell,\ldots,x_{\ell+k}$ is such a substring, then there exists a boundary sequence y starting with the string $x_\ell,\ldots,x_{\ell+k}$, so that $|r(y)-r(\sigma^\ell x)|<\frac{q}{2}3^{-k}$.

We show that if k is sufficiently large, then the growth rate of W_k is close to 3, so that W_k describes a subset of S of dimension close to 1.

The graph G_∂ has $2\lfloor\frac{q-1}{2}\rfloor<q$ vertices; from each vertex v there is at least one word w of length $<\log_3 q$ which does not label a path from v in G_∂. We claim that there is a word γ of length $c < q\log_3 q$ which does not label a path starting at *any* vertex. To see this, for each word β let V_β be the subset of vertices x for which there is a path labelled β starting at x. For the empty word $\emptyset$, $V_\emptyset$ is the set of all vertices. Inductively define γ as follows: suppose an initial segment α_1 of γ is defined with subset V_{α_1}. Take a vertex $v \in V_{\alpha_1}$, and a word β of length $< \log_3 q$ which does not label a path from vertex v. Define α_2 to be the concatenation of α_1 and β. Then $|V_{\alpha_2}| \le |V_{\alpha_1}| - 1$ since it does not contain v. Repeat, defining $\alpha_3, \alpha_4, \ldots$ until there are no vertices in V_{α_ℓ} for some $\ell < q$. Let $\gamma = \alpha_\ell$; then the length of γ is at most $q\log_3 q$, and the word γ does not label any path in G_∂. This proves the claim.

Let $k > 2c$. Then W_k contains all sequences of the form

$$\gamma, w_1, \gamma, w_2, \gamma, \ldots$$

where the w_j are *arbitrary* words of length $k - 2c$, because any subword of length k of $\gamma, w_1, \gamma, w_2, \gamma, \ldots$ contains the subword γ and hence does not label a path in G_∂. So the growth of W_k is at least $\xi = 3^{\frac{k-2c}{k-c}}$, the growth of words of this type. Since $c < q\log_3 q$, we have $\log_3 \xi > 1 - \frac{q\log_3 q}{k - q\log_3 q}$.

If two sequences $a, b \in W_k$ have different first elements $a_1 \neq b_1$, then $|r(a) - r(b)| \ge \frac{q}{2}\cdot 3^{-k}$ (otherwise the initial segments of a and b of length k would both label allowed paths in G_∂).

For each $x \in W_k$ there is a corresponding sequence x' obtained by replacing each occurrence of digit p by the digit qu, so that x' describes a sequence of digits in $\{0, qu, q\}$ and $r(x') \in qS_u$. Note that $|r(x) - r(x')| \le \sum_{i=1}^{\infty} |p - qu| 3^{-i} = |p - qu|/2 < q\epsilon/2$. Let $k = -\lfloor \log_3 3\epsilon \rfloor$, then $|r(x) - r(x')| < q3^{-k}/6$. Let W_k' be the set of such sequences x'.

If $a', b' \in W_k'$ correspond to $a, b \in W_k$ and $a_1' \neq b_1'$ then

$$\begin{aligned} |r(a') - r(b')| &\ge |r(a) - r(b)| - |r(a) - r(a')| - |r(b) - r(b')| \\ &\ge \frac{q}{2}3^{-k} - \frac{q}{6}3^{-k} - \frac{q}{6}3^{-k} \\ &\ge \frac{q3^{-k}}{6}. \end{aligned}$$

So for $n > k$ it takes at least ξ^n intervals of length $q3^{-n}/6$ to cover $r(W_k') \subset S_u$, where ξ is the growth rate of W_k. This implies the Minkowski dimension of S_u (and hence the Hausdorff dimension by [4]) is at least

$$\log_3 \xi \ge 1 - \frac{1}{\frac{k}{q\log_3 q} - 1} > 1 - \frac{1}{\frac{\log 1/\epsilon}{q\log q} - 1}. \qquad \blacksquare$$

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